



## Optimization Problems

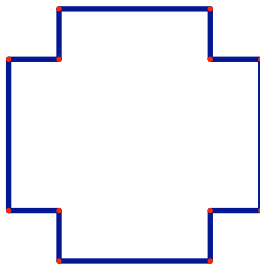
In this section we will apply our skills in identifying maxima and minima to problems in optimization.


**Example**  I have 400 meters of fence with which I will fence off a rectangular plot of land. What should the dimensions of the plot be in order to maximize the area of the plot?

### General principles for solving optimization problems

1. *Read the problem carefully* so that you fully understand what you have to solve for and what information you are given.
2. *Draw a diagram* and identify what information you are given and what you need to find.
3. *Introduce notation* If possible express the unknown quantity to be maximized/minimized as a function of one or two variables. Write down all equations relating these variables.  
Try to express the unknown quantity to be maximized/minimized as a function of one variable.
4. *Identify the critical points of the function* and use these and the values at endpoints to find the absolute maximum/ minimum of the function as appropriate.

**Example**  An open top box is to be made by cutting small congruent squares from the corners of a 12-in by 12-in sheet of cardboard and bending up the sides. Find the largest possible volume of the box.



**Example**  Jane is 2 miles offshore in a boat and wishes to reach a coastal village 6 miles down the straight shoreline from the point nearest the boat. She can row at 2 mph and can walk at 5 mph. Where should she land her boat to reach the village in the least amount of time?

To find absolute extreme values for a function  $f(x)$  on an interval (perhaps  $(0, \infty)$  or  $(-\infty, \infty)$ ), we can sometimes use the following variant of the first derivative test, which agrees with our common sense:

**Modified First Derivative Test** Suppose that  $c$  is a critical number of a continuous function  $f$  on an interval  $I$ . Then

1. If  $f'(x) > 0$  for all  $x < c$  and  $f'(x) < 0$  for all  $x > c$ , then  $f(c)$  is the absolute maximum value of  $f$ .
2. If  $f'(x) < 0$  for all  $x < c$  and  $f'(x) > 0$  for all  $x > c$ , then  $f(c)$  is the absolute minimum value of  $f$ .

**Example** How close does the curve  $y = \sqrt{x}$  come to the point  $(3/2, 0)$ .

**Example** The revenue from selling  $x$  widgets is  $\$6x$  where  $x$  is given in millions. The cost of producing  $x$  items is given by  $c(x) = x^2 + 8$ . Is there a production level that maximizes profit,  $p(x) = r(x) - c(x)$ ?

$$p(x) = r(x) - c(x) = 6x - [x^2 + 8] = -x^2 + 6x - 8 = -[x^2 - 6x + 8].$$

We must maximize  $p(x)$  on the interval  $[0, \infty)$ .

Critical Points of  $p(x)$ :

$$p'(x) = -2x + 6 = -2(x - 3). \quad p'(x) = 0 \text{ if } \boxed{x = 3}.$$

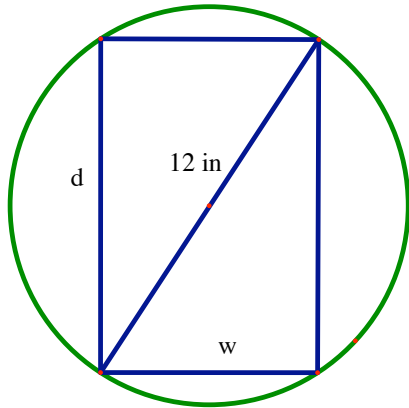
We see that  $p'(x) < 0$  if  $x > 3$  and  $p'(x) > 0$  if  $x < 3$ .

Therefore at  $x = 3$  the profit function reaches a local maximum, by the first derivative test.

Since  $p(x)$  is increasing on the interval  $(0, 3)$  and decreasing on the interval  $(3, \infty)$ , we can conclude that there is a global maximum at  $x = 3$ .

The profit at  $x = 3$  (million widgets) is  $P(3) = -9 + 18 - 8 = 1$  million dollars.

**Extra Example(answer on next page)** The strength  $S$  of a rectangular wooden beam is proportional to its width times the square of its depth. Find the dimensions of the strongest beam that can be cut from a 12 inch diameter cylindrical log.



It is given that  $S = kwd^2$  where  $k$  is a constant. If we find the value of  $w$  that gives us the maximum of the function  $S_1 = wd^2$ , then the same value of  $w$  maximizes  $S$ .

So we want to maximize  $S_1 = wd^2$  for  $0 \leq w \leq 12$ .

We also have  $d^2 + w^2 = 144$ , giving us that  $d^2 = (144 - w^2)$ .

This gives us that  $S_1 = w(144 - w^2) = 144w - w^3$ .

Critical Points:

$$\frac{dS_1}{dw} = 144 - 3w^2.$$

$$\frac{dS_1}{dw} = 0 \text{ if } 144 = 3w^2 \text{ or } w = \sqrt{48},$$

since  $w$  must be greater than 0.

We know that  $S_1$  has an absolute maximum at  $w = \sqrt{48}$  from checking the value of  $S_1 = w(144 - w^2)$  at the endpoints of the interval  $[0, 12]$  and at the critical point  $w = \sqrt{48}$ .

$$S_1(0) = 0, \quad S_1(\sqrt{48}) = \sqrt{48}(144 - 48) = 96\sqrt{48}, \quad S_1(12) = 12(144 - 144) = 0$$